SPR and ASPR Untangled

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Abstract: The theory, proofs, use and implementation of robust adaptive control algorithms require the understanding of the concepts of Strictly Positive Real (SPR) and Almost Strictly Positive Real (ASPR) plants. Although these concepts are defined in the existing literature, both in time and frequency domains, their grasp is not straight forward for the practicing control engineer that deals with real-world plants. Here, we attempt to present the interpretation and meaning of these concepts in a more intuitive way for the practicing control engineer. That is, we use the Bode, Nyquist, Nichols and Root-Locus domains that may help the control engineer better grasp their implications. The paper is oriented towards the practitioner and therefore any proof that already appears in the literature is only cited. An important result that is formalized and proved is that any stable system can be made ASPR. An example is also given in order to demonstrate these concepts.

1. INTRODUCTION

When tackling challenging control problems, one often considers the application of adaptive control. This may be a Cost-Effective approach if it does not require design of new hardware.
Application of adaptive control in general and the Simplified Adaptive Control (SAC) [1, 2] in particular require the control system to be Almost Strictly Positive Real (ASPR).
Recent work [7] shows how the performance of any stable existing servo system can be improved by applying an Add-On SAC based controller. To emphasize it is shown in [7] that for a stable system there always exists a controller that improves the system's performance. This approach guarantees improvement in performance and improvement in robustness of performance under changing operating regime.
However, the implementation, understanding and proofs of performance of the Add-On adaptive controller require the understanding of the concepts of SPR and ASPR. Therefore, this paper reviews the SPR and ASPR and presents their interpretation in the Bode, Nyquist, Nichols and Root-Locus domains. This helps the control engineer better grasp their implications.
It is shown (and proved) that for any proper stable system there always exist an algorithm – the so-called Parallel Feedforward Configuration (PFC) - that makes the plant ASPR, thus showing that the performance of any proper stable system can be improved by the Add-On SAC based algorithm. The proof is constructive and it is very simple to derive the required PFC.

2. SPR UNTANGLED

In this section, we first present the rigorous definition of SPR system in time and frequency domain. Then, intuitive interpretations present, for the engineers, the SPR concept in each of the engineering descriptions: Bode, Nyquist, Nichols and Root-Locus domains.
Let the SISO linear system be strictly proper system and represented by

$$\begin{aligned}\dot{\boldsymbol{x}}_p(t) &= \boldsymbol{A}_p\boldsymbol{x}_p(t) + \boldsymbol{B}_p\boldsymbol{u}_p(t)\\ \boldsymbol{y}_p(t) &= \boldsymbol{C}_p\boldsymbol{x}_p(t)\end{aligned} \tag{1}$$

2.1. Definition of SPR in time domain [2,4]

Definition 1: The SISO strictly proper minimal realization (12)-is called "strictly positive real" (SPR) if the following time-domain relations are satisfied:

$$\begin{aligned}&\exists\, \boldsymbol{P},\boldsymbol{Q} > 0, \textit{such that}\\ &\boldsymbol{P}\boldsymbol{A}_p + \boldsymbol{A}_p^T\boldsymbol{P} = -\boldsymbol{Q} < 0\\ &\boldsymbol{P}\boldsymbol{B}_p = \boldsymbol{C}_p^T;\end{aligned} \tag{2}$$

2.2. Definition of SPR in frequency domain [2]

Definition 2: The transfer function Z(s) is **SPR** if and only if
1. **Z**(s) is stable (does not have poles in the closed RHP);
2. Re{**Z**(jω)}>0 for all |ω|<∞;
3. **Z**(s) is real for real s (any physical system satisfies this condition);
4. $\lim_{\omega\to\infty} \mathrm{Re}\{\omega^2 \boldsymbol{Z}(j\omega)\} > 0$.

Notice: For any practical purpose conditions 1 and 2 are the major conditions from engineering point of view. Conditions 3 and 4 are presented here for the rigorous definition.

Fact 1: G(s) is SPR ⇔ $G^{-1}(s)$ SPR.
Fact 2: SPR ⇒stable + minimum phase + relative degree = r ∈{-1, 0, 1}.

Fact 3: The converse of fact 2 is not correct, i.e. stable + minimum phase + relative degree = r ∈{-1, 0, 1} are necessary but not sufficient requirements.

2.3 Engineering understanding:

If the system Z(s) is SPR, this implies that the system remains stable for any positive static gain, even arbitrarily large, i.e. Z(s)/(1+kZ(s)) is stable for any k≥0. This means that such system can achieve any closed loop bandwidth. However, real SPR systems are rare, to say the least.
The SPR requirement means that the transfer function is such that the phase of the transfer function is $-90\deg < \angle \boldsymbol{Z}(j\omega) < 90\deg$, and also, simultaneously that the slope of the Bode plot is $-20\boldsymbol{dB}/\boldsymbol{dec} < |\boldsymbol{Z}(j\omega)| < 20\boldsymbol{dB}/\boldsymbol{dec}$.
The full interpretation of SPR in Bode, Nyquist and Nichols domains is complicated and can not be put clearly in a single figure. The following are strictly sufficient conditions for SPRness in those domains. We sacrifice some comprehensiveness for simplicity.

2.4 SPR in Bode Domain

Figure 1 presents interpretation of SPR in Bode Domain. SPR means that the Bode plot can not roll down faster that at rate of -20dB/dec.

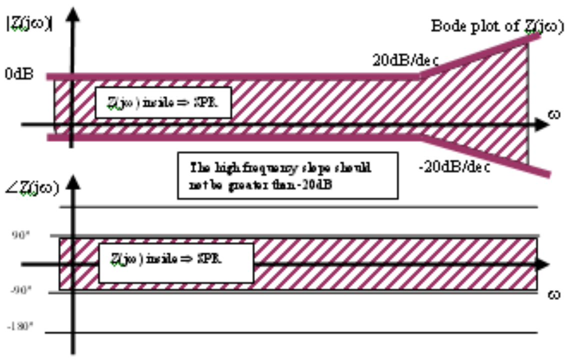

Figure 1: Interpretation of SPR condition in Bode domain.

2.5 SPR in Nyquist Domain

Figure 2 presents the interpretation of SPR in Nyquist Domain. The requirement on the phase means that the graph of the transfer function should lie in the open LHP of the Nyquist domain-the complex plane.

2.6 SPR in Nichols Domain

Figure 3 presents the interpretation of SPR in Nichols domain. The requirement on the phase means that the graph of the transfer function phase should lie between -90deg and +90deg.

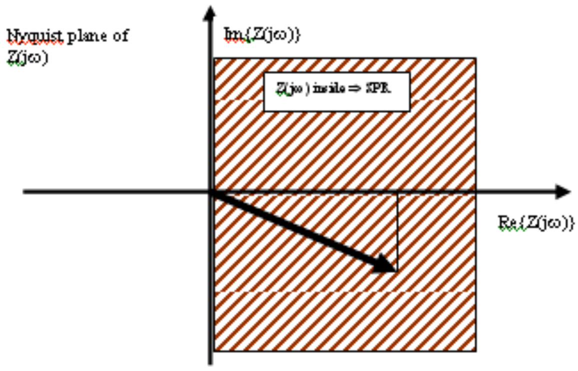

Figure 2: Interpretation of SPR condition in Nyquist domain.

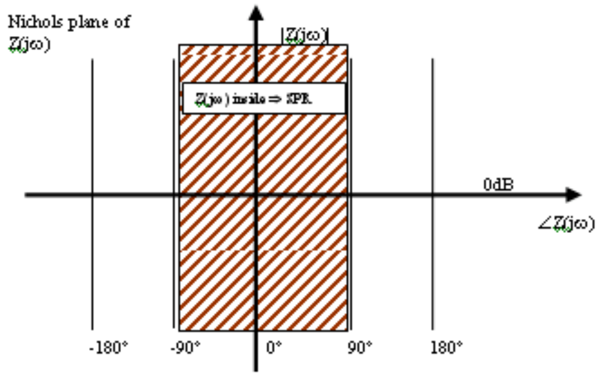

Figure 3: Interpretation of SPR condition in Nichols domain.

2.7 SPR in Root-Locus

To check that a given transfer function is SPR in Root-Locus domain, i.e. from pole-zero location in the complex plane, is complicated. At an engineering level the best that can be said today is:
The poles and zeros should be sufficiently distant to the left of the imaginary axis and pole–zero pairs should be sufficiently close one to another. An example of SPR plant pole-zero location is shown in Figure 4.

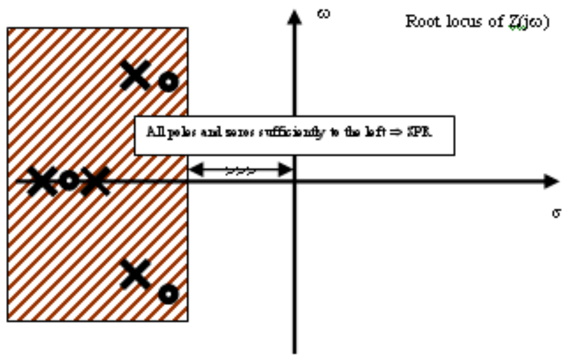

Figure 4: Interpretation of SPR condition in Root-Locus domain.

3. ASPR UNTANGLED FOR STABLE SYSTEMS

Because, as we mentioned, SPR is a very strong property, not necessarily satisfied in real-world, [2] has introduced the concept of "Almost" SPR (ASPR).
In this section we first present the rigorous definition of the ASPR system in time and frequency domains for general

system. Then, we also present for stable systems an intuitive interpretation of the ASPR concept for the engineers in each of the engineering descriptions: Bode, Nyquist, Nichols and Root-Locus domains.

3.1. Definition of ASPR in time domain [1,2]

Definition 3: Let the be strictly proper SISO linear system be represented by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3}$$

Then if there exists a gain K_e such that the closed loop system

$$\begin{aligned} \dot{x}(t) &= (A - BK_eC)x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{4}$$

is SPR, the original open loop system (1) is called ASPR.

3.2. Definition of ASPR in frequency domain [1,2]

Definition 4: Let $P(s)$ be a transfer function. If there exist a constant gain, K_e, not necessarily known, such that the closed loop transfer function

$$Z(s) = \frac{P(s)}{1 + K_eP(s)} \tag{5}$$

is SPR, then $P(s)$ as called Almost Strictly Positive Real (ASPR).

Lemma 1 [1]**:** Let **Z**(s) be a SISO minimum phase transfer function of relative degree 1 (n poles and n-1 zeros). Then, Z(s) is ASPR.
In other words, the system $P(s)$ is ASPR if, for some $K_e > 0$, the closed loop system $Z(s) = P(s)/(1 + K_eP(s))$ is SPR.
The above means, and interpreted in the Bode, Nyquist, Nichols and Root-Locus domains, that ASPR systems are stabilizable by some positive gain (and remain stable even for very large gains).

The full interpretation of ASPR in Bode, Nyquist and Nichols domains is complicated and can not be put clearly in a single figure. The following are **strictly** sufficient conditions for ASPRness in those domains for stable systems. We sacrifice some comprehensiveness for simplicity.
At an engineering level, for stable systems (that are dealt with here and in the co-paper [7]), we have:
The ASPR requirement means that the transfer function is such that the phase of the transfer function is at low frequencies should not be too high, and at high frequencies $-90\,\deg < \angle Z(j\omega) < 90\,\deg$. This means, simultaneously, also that the slope of the Bode plot is at low frequencies can not be high and at high frequencies we require $-20dB/dec <$ slope of $|Z(j\omega)| < 20dB/dec$.

3.3 ASPR in Bode Domain

Figure 5 presents the interpretation ofASPR in Bode Domain. For proper systems this means that the Bode plot can not: (i) at low frequencies, roll down (up) faster that -40dB/dec(40dB/dec); and (ii) at high frequencies, roll down (up) faster that at rate of -20dB/dec (20dB/dec).(Recall that the Bode Integral connects the absolute value and the phase of the Bode plot.)

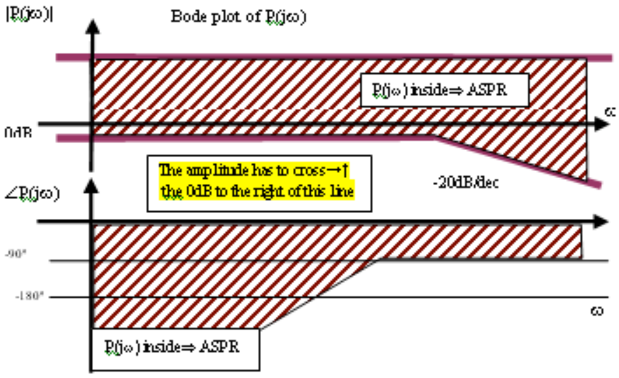

Figure 5: Interpretation of ASPR condition in Bode domain.

3.4 ASPR in Nyquist Domain

Figure 6 presents the interpretation of ASPR in Nyquist Domain. It means that the Nyquist plot exclude the circle of radius 1/2k, k>0, with origin at [-1/2k,0], i.e. arrive at the origin with phase of 90deg, and should not encircle the at [-1/k,0] point.

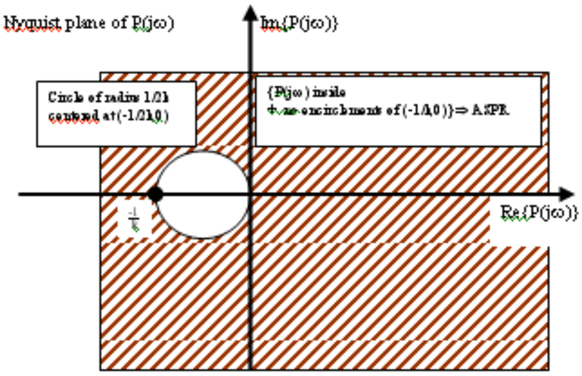

Figure 6: Interpretation of ASPR condition in Nyquist domain.

For constant gain it would be encircle the at [-1/k,0] point for stability in closed loop for any constant gain, k. However since in adaptive control the gains are time varying the additional condition (exclusion of the circle and arrival at -90 deg) arises.
3.5 ASPR in Nichols Domain

Figure 7 presents the interpretation of ASPR in Nichols domain. It is a sufficient condition.

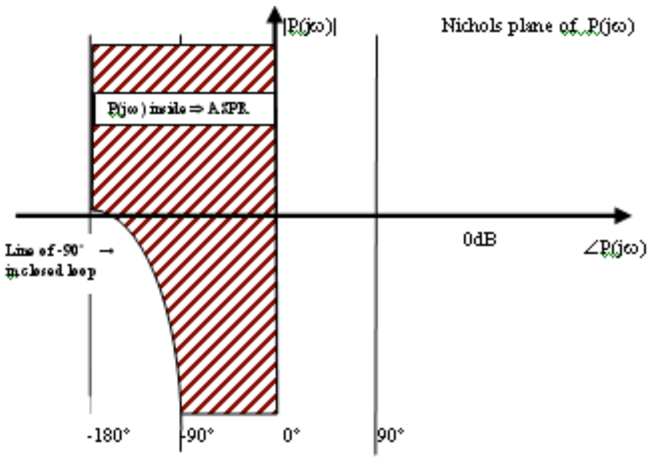

Figure 7: Interpretation of ASPR condition in Nichols domain.

3.6 ASPR in Root-Locus

To check that a given transfer function is ASPR in Root-Locus domain, is simple. The requirement is that there is at most one asymptote, and for proper systems this means exactly one asymptote.
An example is shown in Figure 8. One can see that that for sufficiently large gain, n-1 poles converge to the zeros and one pole goes to minus infinity. Thus, the total angle of Z(jω) remains within +-90deg.

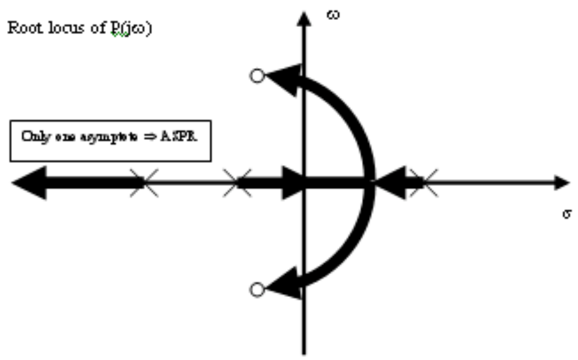

Figure 8: Interpretation of ASPR condition in Root-Locus domain.

4. PARAMETERIZATION OF PFC THAT RENDERS STABLE PLANTS ASPR

Here we deal with the following issue: when can a plant be "converted" into an ASPR plant? For convenience, we only deal here with stable systems. The main result that is formalized and proved in this section is that any stable system can be augmented in such a way that it is made ASPR.
We will parameterize a set of parallel feed-forward configurations (PFC) that convert any stable plant into an ASPR plant. This shows that any stable plant is "ASPRable."

4.1 Definitions

We deal with transfer functions that are not necessarily proper rational functions. For the sake and convenience of the mathematical generality, we define as follows:
1. A rational (not necessarily proper) function is stable if all its poles are in the **open** LHP.
2. A rational function is minimum phase if all its zeros are in the **open** LHP (else it is non-minimum phase).
3. Relative degree, r: r = degree of the denominator – degree of the numerator.
4. $\tilde{\mathbf{S}}$ Family of all stable, not necessarily proper, real-rational functions.
5. $\mathbf{S}$ Family of all stable, proper, real-rational functions.
6. $\tilde{S}_r, S_r$ The above families restricted to relative degree r.

In this work we assume that the plant P(s) :
(i) is stable;
(ii) is strictly proper, i.e. the relative degree = $r \geq 1$;
(iii) has finite DC gain. $0 \leq P(0) < \infty$;

We use ideas from the following theorem from [5]:
Theorem 1: Assume that the system $\boldsymbol{P} \in \boldsymbol{S}$ (stable *and* proper). The set of all controllers, $\boldsymbol{C}$, for which the feedback system is internally stable equals

$$\left\{ \frac{\boldsymbol{Q}}{\boldsymbol{1\text{-}QP}} ; \boldsymbol{Q} \in \boldsymbol{S} \right\} \tag{6}$$

Proof: The full proof is presented in [5]. Here we present only a sketch. If both Q and P are stable, then

$$\frac{\boldsymbol{CP}}{1+\boldsymbol{CP}} = \frac{\frac{\boldsymbol{Q}}{1-\boldsymbol{QP}}\boldsymbol{P}}{1+\frac{\boldsymbol{Q}}{1-\boldsymbol{QP}}\boldsymbol{P}} = \frac{\boldsymbol{QP}}{1-\boldsymbol{QP}+\boldsymbol{QP}} = \boldsymbol{QP} \in \boldsymbol{S} \tag{7}$$

Q.E.D.

Notice:
i) $\mathbf{C}$ in theorem 1 is not necessarily stable or minimum phase!
ii) in [5] one can find more general parameterizations of the set of all stabilizing controller for any proper plant.

Theorem 2 [1, 8]**:**
ASPR ⇔ minimum phase + relative degree$\in\{0,1\}$.

Lemma 1:
Assume that the system $\boldsymbol{P} \in \boldsymbol{S}$ (stable and proper). Assume that $\boldsymbol{D}^{-1} = \frac{\boldsymbol{Q}}{1-\boldsymbol{QP}}$ and $\boldsymbol{Q} \in \tilde{\boldsymbol{S}}$ (stable), i.e. $\boldsymbol{D}^{-1}$ stabilizes P .
Then, $\boldsymbol{P}+\boldsymbol{D}$ is minimum phase.
Proof: We have

$$\boldsymbol{P}+\boldsymbol{D} = \boldsymbol{P} + \frac{\boldsymbol{1\text{-}QP}}{\boldsymbol{Q}} = \frac{\boldsymbol{QP}+\boldsymbol{1\text{-}QP}}{\boldsymbol{Q}} = \frac{\boldsymbol{1}}{\boldsymbol{Q}}, \tag{8}$$

as $\boldsymbol{Q} \in \tilde{\boldsymbol{S}}$ (stable) then $\boldsymbol{Q}^{-1}$ is minimum phase. Q.E.D.

Theorem 3: Assume that the system $\boldsymbol{P} \in \boldsymbol{S}$ (stable+ proper). Let $Q \in \tilde{S}_{-1}$ (i.e. stable+relative_degree=–1}).

If $D^{-1} = \frac{Q}{1-QP}$ then $P_a(s) = P + D$ is minimum_phase + relative_degree=1 and thus ASPR.
Proof: We have

$$P_a(s) = P + D = P + \frac{1\text{-}QP}{Q} = \frac{QP + 1\text{-}QP}{Q} = \frac{1}{Q}. \qquad (9)$$

$Q \in \tilde{S}_{-1}$ (stable+relative_degree r=−1) therefore, Q^{-1} is minimum phase and of relative degree r=1. Therefore, from theorem 2, Q^{-1} it is ASPR. Q.E.D.

Theorem 3a: Assume that the system $P \in S$ (stable+ proper). Let Q^{-1} be minimum phase + relative_ degree=−1). If $D(s) = Q^{-1}(s) - P(s)$, then $P_a(s) = P(s) + D(s)$ is minimum phase + relative_degree=1 and thus ASPR.
Proof: We have
$P_a(s) = P + D = Q^{-1}$, if $Q \in \tilde{S}_{-1}$. Q^{-1} minimum phase and relative degree r=1. Therefore from theorem 2 it is ASPR.. Q.E.D.

Remarks

(i) What we called $D(s)$ actually is $D(s) = C^{-1}(s)$ from theorem 1.
(ii) The converse of theorem 3 is not correct. That is, theorem 3 does not parameterize all PFC's that are ASPRing stable plants.
(iii) Theorem 3 states that any stable plant is ASPRable by parallel feed forward transfer function of relative degree 1. (Note that we only restricted Theorem 3 to stable systems because of its simplicity and its direct applicability to many real world problems. However, in general, ASPR plants that are created by parallel feed forward do not require the original plant to be stable.)
(iv) Although tempting, $Q \not\subset \{SPR \wedge r = 1\}$, Q is not subset of SPR transfer functions with relative degree one.
(v) More general parameterization of PFC transfer function can be derived by procedures of coprime factorization [5].

5. Example: Design of PFC

A great amount of examples can be found in the literature (see [2] and references therein). In this example we assume.

$$k_2 HG(s) = \frac{s+4}{s^3 + 5s^2 + 6s} \qquad (10)$$

Then

$$P(s) = \frac{k_2 HG(s)}{1 + k_2 HG(s)} = \frac{s+4}{s^3 + 5s^2 + 7s + 4} \qquad (11)$$

We select a minimum phase transfer function with relative-degree=-1

$$Q(s) = \frac{25(s+0.5)(s^3 + 5s^2 + 7s + 4)}{s^3 + 30s^2 + 119.5s + 55} \qquad (12)$$

Then from theorem 3

$$D(s) = Q^{-1}(s) - P(s) = \frac{1}{25(s+0.5)} \qquad (13)$$

We need to implement only $D(s)$ in (8,9), and we have the augmented plant (9)

$$P_a(s) = P(s) + D(s) = \frac{1}{Q(s)} = \frac{s^3 + 30s^2 + 119.5s + 54}{25(s+0.5)(s^3 + 5s^2 + 7s + 4)} \qquad (14)$$

Notice that except one pole at s=-0.5 the rest of the poles and zeros are close respectively. Consequently, the augmented plant $P_a(s) = P(s) + D(s)$ is stable for any positive gain. From figure 9, one can see that at low frequencies the relative effect of the PFC is sufficiently small such that $P(s)$ and $P_a(s)$ practically coincide. The result is an augmented plant which is minimum-phase, with 4 poles and three finite zeros, so it is ASPR and adaptive control can be applied.

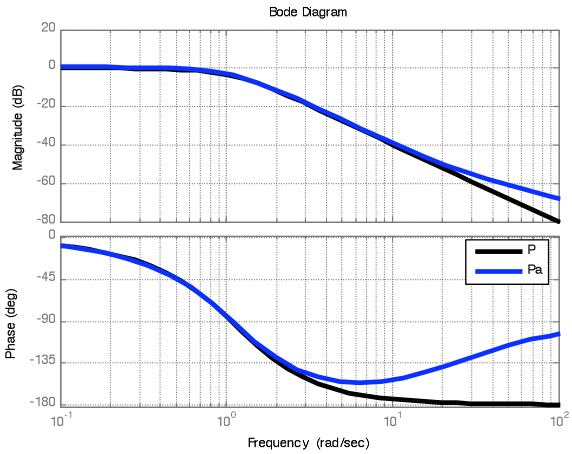

Figure 9: The Bode plot of the original plant, P(s), and the augmented plant, $P_a(s)$.

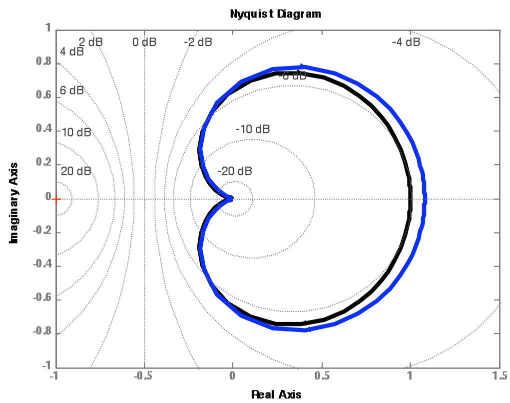

Figure 10: The Nyquist plot of the augmented plant – does not cross the negative real axis.

Figures 10 and 11 present the Nyquist plot of the ASPRed plant $P_a(s)$. One can see from figure 11, a zoom up of figure 10 around the origin, that the Nyquist plot arrives (leaves) at phase of -90deg (90deg), respectively.

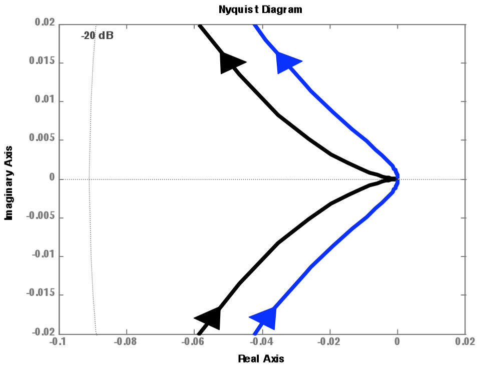

Figure 11: The Nyquist plot of the augmented plant zoomed near the origin – reaches the origin from ±90deg directions.

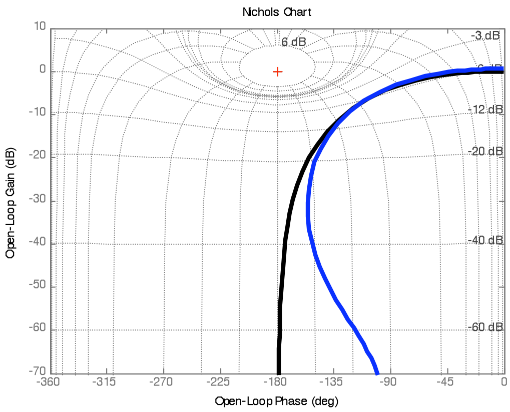

Figure 12: The Nichols plot of the augmented plant – does not deviate beyond -180deg at low frequency and -90deg at high frequencies.

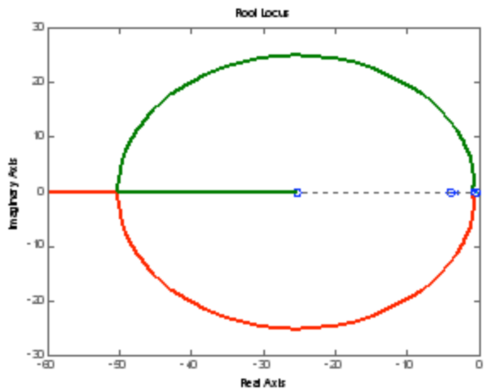

Figure 13: The Root Locus of the augmented plant – has only one asymptote.

Figure 12 presents the Nichols plot. One can see that all sufficient conditions are satisfied.
Figure 13 presents the interpretation of ASPR in the Root-Locus domain.

6. CONCLUSIONS

The interpretation and meaning of the Strictly Positive Real (SPR) and Almost Strictly Positive Real (ASPR) concepts in the Bode, Nyquist, Nichols and Root-Locus domains is presented. This helps the practicing control engineer more intuitive and better grasp of their implications.
It is proved that any stable plant can be augmented to become ASPR.

7. REFERENCES

1] Bar-Kana, I.: "Parallel Feedforward and Simplified Adaptive Control," International Journal of Adaptive Control and Signal Processing, Vol. 1, No. 2, 1987, pp. 95-109.

2] Kaufman, H., Barkana, I., and Sobel, K.S.: Direct Adaptive Control Algorithms-Theory and Applications, Springer, New York, 1988 (2nd edition).

4] Anderson, B.D. O, and Vongpanitlerd, 1973, Network Analysis and Synthesis, Prentice Hall, Englewood Cliffs, NJ.

5] Doyle, J.C., Francis, B.A. and Tannenbaum, A.R: Feedback Control Problems, Macmillan Publishing Company, 1992.

6] Weiss, H., Wang, Q. and Speyer, J.L.: "System Characterization of Positive Real Conditions," IEEE Transactions, on Automatic Control, Vol. 39, No. 3, March 1994, pp. 540-544.

7] Rusnak, I. and Barkana, I.: Improving Performance of Servo Systems using Adaptive Control. (in preparation)

8] Bar-Kana, I.: "Comments on 'Design of Strictly Positive Real Systems Using Constant Output Feedback'," IEEE Transactions on Automatic Control, Vol. TAC-49, 2004, pp. 2091-2093.